The impact of Common Core Standards on the mathematics education of teachers

MAA Section Meeting

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In June of 2010, the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) jointly released a set of **Common Core Standards** for mathematics and reading, and asked the states for voluntary adoption.

Three states—Alaska, Minnesota, and Texas—rejected these standards, but over 40 other states have signed on in the meantime. Implementation will take place in 2014.

This talk is about the Common Core Mathematics Standards (**CCMS**).

As of 2011, almost every state has its own set of math standards, and *they are all different from each other in some ways*. The differences are damaging to students' math education; they also obstruct any attempt to improve school math textbooks.

The need of a good set of common standards is real.

But is CCMS good enough?

CCMS has its flaws— that is inevitable—but its mathematical quality is, overall, far superior to the existing state standards.

- Does not play the game of educational one-upmanship: does not try to teach each topic earlier than other standards.

- Main emphases:

  (a) Restore some mathematical clarity and precision to school mathematics.

  (b) Maintain logical continuity from grade to grade, and infuse reasoning in the presentation of each topic.

What stands in the way of a successful implementation of CCMS?

Textbooks

Assessment

Teacher Quality

As of 2011:

Quality of math textbooks: Very poor in general.

Quality of assessment: Poor in general.

Teachers' content knowledge (of mathematics): Fragile, due to reasons to be discussed.

How to get adequate math textbooks to support CCMS is a major, major issue.

Textbook publishers are driven by one thing only: the bottom line. This translates into, *not* exposition of higher mathematical quality, but more adoptable textbooks, i.e., books that most teachers feel are easy to use, which is distinct from books that make more mathematical sense.

If you know how to deal with the bottom-line mentality in education, please call me 24/7.

The problem with state assessments deserves to be discussed at length, and all by itself.

There is a discussion of current state assessments in the *National Mathematics Advisory Panel Report, 2008.*

The way the Common Core Assessment is shaping up, there is ample room for concern: Are there *knowledgeable* mathematicians involved to do quality control? Will students be over-tested?

Every state should be on full alert.

How to get mathematically knowledgeable teachers to implement CCMS is the main subject of this talk.

The problem is every bit as intractable as the other problems, but it is something over which we academics have *some* control.

Our goal is to help produce teachers who are proficient in **school mathematics (SM)**.

By *SM*, we mean the mathematics of the standard school math curriculum:

$$\text{whole numbers} \longrightarrow \left\{ \begin{array}{c} \text{fractions} \\ \text{integers} \end{array} \right\} \longrightarrow \text{rational numbers}$$

$$\longrightarrow \left\{ \begin{array}{c} \text{algebra} \\ \text{geometry} \end{array} \right\} \longrightarrow \text{trigonometry and pre-calculus}$$

Of course one may add to this a small amount of statistics.

A university mathematician's typical reaction to school mathematics:

> it is elementary,
>
> therefore trivial,
>
> therefore if we teach future teachers "real" mathematics "the right way", they will understand this elementary stuff.

The mistake was made in a disastrous way in the New Math era.

Fundamental Fact:

Much of SM is not part of university mathematics.

We can see this by considering a typical problem in SM:

Convert $\frac{5}{27}$ to a decimal with 6 decimal digits.

$$\frac{5}{27} = \frac{5 \times 10^6}{27 \times 10^6} \qquad \text{(equivalent fractions)}$$

$$= \frac{5 \times 10^6}{27} \times \frac{1}{10^6} \qquad \text{(product formula: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\text{)}$$

Caution: We want the long division of 5×10^6 (= 5,000,000) by 27, but the fraction $\frac{5 \times 10^6}{27}$ itself is not the long division of 5,000,000 by 27.

$$
\begin{array}{rrrrrrrrr}
 & & & 1 & 8 & 5 & 1 & 8 & 5 \\ \cline{2-9}
27 &) & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 2 & 7 & & & & & \\ \cline{3-5}
 & & 2 & 3 & 0 & & & & \\
 & & 2 & 1 & 6 & & & & \\ \cline{3-6}
 & & & 1 & 4 & 0 & & & \\
 & & & 1 & 3 & 5 & & & \\ \cline{4-7}
 & & & & & 5 & 0 & & \\
 & & & & & 2 & 7 & & \\ \cline{5-8}
 & & & & & 2 & 3 & 0 &
\end{array}
$$

We obtain: The long division of 5,000,000 by 27 has quotient 185185 and remainder 5.

School textbooks say:

$$5,000,000 \ = \ 185185 \text{ R } 5$$

This doesn't make sense.

The correct symbolic expression is:

$$5,000,000 \ = \ (185185 \times 27) + 5$$

$$
\begin{aligned}
\frac{5}{27} &= \frac{5 \times 10^6}{27 \times 10^6} && \text{(equivalent fractions)} \\
&= \frac{5 \times 10^6}{27} \times \frac{1}{10^6} && \text{(product formula: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\text{)} \\
&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6}
\end{aligned}
$$

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&= \left(185185 + \frac{5}{27}\right) \times \frac{1}{10^6} \\
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&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6} \\
&= \left(185185 + \frac{5}{27}\right) \times \frac{1}{10^6} \\
&= \frac{185185}{10^6} + \left(\frac{5}{27} \times \frac{1}{10^6}\right) \\
&= 0.185185 + \left(\text{a positive number} < \frac{1}{10^6}\right)
\end{aligned}
$$

Some observations:

(A) This is a basic topic in SM, but it does not sit comfortably in any standard university-level math course.

(B) **SM is different from university mathematics:**

> In SM, the product formula is not a definition of fraction multiplication, but a nontrivial **theorem**.

(C) The reasoning behind the conversion:

requires an understanding of what long division means,

shows that this mysterious sounding conversion is nothing more than expressing a given fraction as another fraction with denominator equal to 10^6, and

shows the conversion to be a consequence of the product formula in *fraction* multiplication.

How is this conversion taught in school textbooks, and therefore in school classrooms?

A rote skill that makes the long division algorithm even more mystifying:

			.1	8	5	1	8	5
27	)	5	.0	0	0	0	0	0
		2	7					
		2	3	0				
		2	1	6				
			1	4	0			
			1	3	5			
					5	0		
					2	7		
					2	3	0	

A set of state standards calls for the conversion in 7th grade: "Know that every rational number can be written as the ratio of two integers or as a terminating or repeating decimal."

For sure the rote procedure will be paraded in all the 7th grade classrooms of that state.

Yet this standard will be cited as an example of *rigor and clarity*. This is the intellectual climate we are in.

We recognize the rote skill as a neat summary of the multistep reasoning, but when the reasoning is suppressed—as in school textbooks and standard educational materials,—it becomes totally incomprehensible because we have

a fraction $\frac{5}{27}$ (which is a piece of pizza),

a decimal $0.185185\ldots$ (a senseless sequence of digits created by long division).

In what sense are they equal, and why?

All of us were taught such rote procedures, and some survived to learn more mathematics. Many others were probably not so fortunate.

The mathematics in school math textbooks is as incomprehensible as this rote procedure about 40 to 50 percent of the time. This is a perversion of SM, and we shall call it

Textbook School Mathematics (**TSM**).

It has been said, with ample justification, that *TSM is our de facto national curriculum.*

Let it be said here that we should get out of **this** curriculum as fast as we can.

To summarize:

SM: It is elementary, perfectly understandable, and not trivial.

TSM: It is irrational, and therefore incomprehensible about half the time.

If our schools can teach SM, then we would have no *Mathematics Education Crisis*. But almost all schools teach TSM.

Why?

Let us follow the *life cycle of school teachers* from the time they were students:

They learned TSM as students in K-12.

$\longrightarrow$ They learn university mathematics in college, but not SM.

$\longrightarrow$ They must fall back on the TSM they learned in K-12 when they become teachers.

$\longrightarrow$ Their students learn TSM from them.

$\longrightarrow$ The next generation of teachers only know TSM.

Mathematics educators are themselves victims of TSM.

The above *life cycle of school teachers* is equally applicable to educators: TSM gets recycled among educators as well.

The only hope of breaking this vicious cycle is to **teach future teachers and educators SM in universities**.

This sounds simple, but there is campus politics. There is also the lack of a de fault version of SM:

> There is, as yet, no systematic exposition of K-12 mathematics that meets
>
> the needs of the school classroom and
>
> the minimum requirements of mathematics
>
> in terms of clarity, precision, reasoning, and cohesiveness.

But the most serious missing component is the contributions of truly competent mathematicians who want to improve school math education and possess the requisite knowledge of schools.

This knowledge is best illustrated by the recognition that since fractions in SM are taught to ten-year olds, its mathematical development

> must be sensitive to their knowledge base and mathematical sophistication, and
>
> must differ significantly from that in university algebra courses.

It would be fair to say that we mathematicians have had a dismal record in educating teachers.

Our efforts have produced books that range from **overly formal and inappropriate** for use by teachers, to **mathematically oversimplistic** in trying too hard to be *pedagogically correct*.

As an example of the latter, there is a volume, *The Mathematical Education of Teachers* **(MET)**, written under the auspices of CBMS. This is the standard reference for the professional development of mathematics teachers.

The authors are a mix of mathematicians and educators, which could in principle produce something that is balanced and beneficial to teachers. However, except for the broad recommendations on the need of more mathematics courses, its detailed guidance—on the whole—falls far short of the ideal.

Because it is being revised for a second edition, CBMS must be aware of the problems and certainly wouldn't mind it if we look at an example of where things go wrong.

Rigid motions, **symmetry**, and **congruence**

are a staple of the middle school curriculum nationwide. These concepts are casually brought up and nonchalantly discarded in the middle school classroom.

Congruence is "same size and same shape", and symmetry is for appreciating beauty in art.

Mathematics cannot be done on this basis.

What is missing is mathematical guidance on

how to delineate their logical interrelationship, and
how to bring out their relevance in mathematics.

Page 33 of MET: "The study of rigid motions can lead to an understanding of congruence . . . Geometry should also be studied as it occurs outside of mathematics, such as in nature and in art. There are many examples that could be studied, such as in the artwork of various cultures (*examples omitted*). Geometric transformations can be found in many designs, and recognizing these transformations adds, for prospective teachers, a legitimacy to the study of transformation by middle grades students."

Page 111: "A careful study of the meaning of congruence and of how congruence can be established should be included."

Two salient points made by MET:

> The justification for the study of rigid motion and transformation is *not a mathematical one*, but must be sought in art and nature.
>
> The "meaning" or "understanding" of congruence that MET has in mind is left to the reader.

Mathematical questions left unanswered:

What is the *mathematical* connection between "same size same shape" and SAS, ASA, SSS in high school?

What is meant by two *curved* geometric figures being *congruent*?

Are rigid motions related to congruence?

Congruence and rigid motions *are* important mathematical topics in SM, but their importance is completely hidden in TSM.

MET seems to be unaware of the disconnect between SM and TSM, and therefore sees no need to offer help where help is urgently needed.

(Writers of state standards also seem to be completely unaware of this disconnect, as are many commentators on CCMS.)

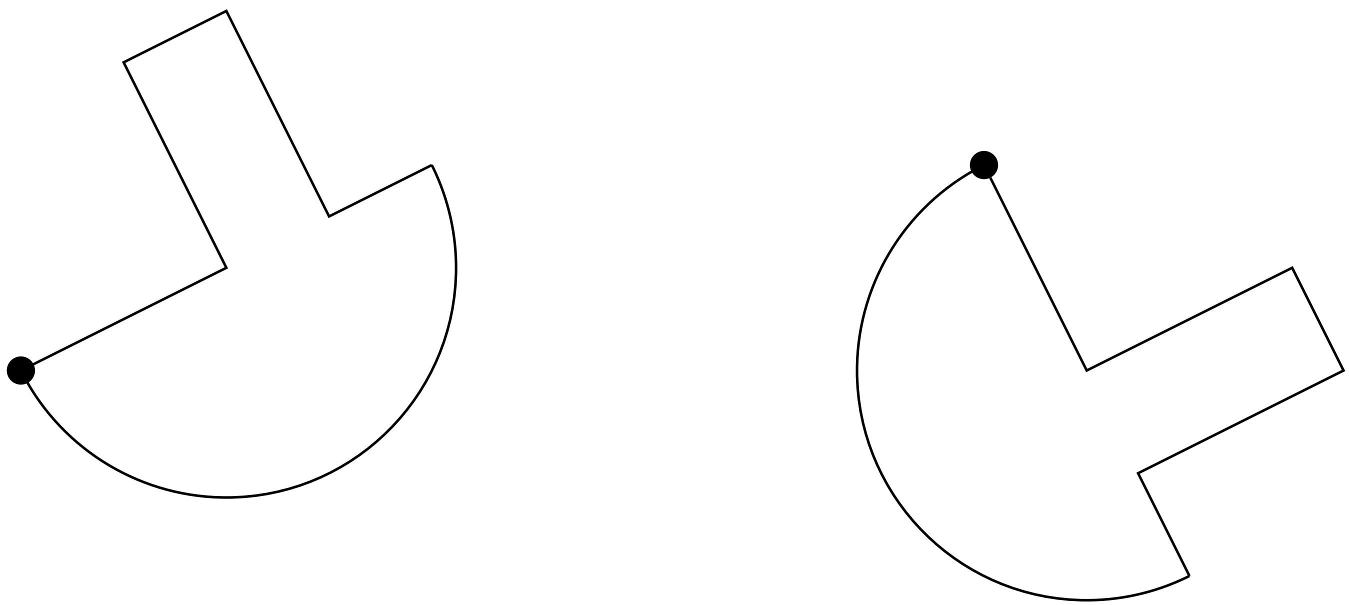
Here is how CCMS tries to align the teaching of geometry in grade 8 and high school with SM:

Grade 8

(1) Introduce the three basic rigid motions—**translation, reflection, rotation**—by hands-on activities, allowing students to gain an *intuitive* understanding of these concepts.

(2) Introduce the concept of the **composition** of basic rigid motions, again by hands-on activities.

(3) Define **congruence** as the composition of a finite number basic rigid motions, emphasizing that congruent figures are intuitively "the same size and same shape". For example:



(4) Prove ASA and SAS for triangles by hands-on activities.

High School

(1) Define **transformation** of the plane.

(2) Define **translation, reflection, rotation** (basic rigid motions) as specific transformations.

(3) Define the **composition** of transformations.

(4) Define **congruence** as the composition of a finite number of basic rigid motions.

(5) Make explicit the *assumption* that basic rigid motions map lines to lines and segments to segments, and are length-preserving and degree-preserving transformations.

(6) Prove ASA, SAS, and SSS as *theorems* about triangles.

The development of plane geometry can essentially proceed as usual at this point.

CCMS gives students the opportunity to see that, beyond art and nature, congruence serves a serious mathematical purpose, and the ASA, SAS, SSS criteria for triangle congruence are more than rote skills. They are an integral part of the fabric that we call *geometry*.

More importantly, CCMS lets students see a different view of the Euclidean plane, one that gets at the geometric essence of the plane:

The plane is what it is, precisely because it possesses these three kinds of basic rigid motions.

CCMS thus tries to make sense of school geometry for students.

It tries to change TSM to SM.

It also exhibits so-called **sense making** in the context of serious mathematics, and not as a slogan:

> Sense making has to begin at the most basic level of the curriculum, and should not be a separate headline in mathematics education.

The last comment about sense making is very germane to our task at hand:

> How can we produce a corps of teachers that understand the core message of CCMS,
>
> make sense of TSM, and
>
> can transform TSM into SM?

CCSM can say all it wants, but without teachers in the classroom to implement its vision, it is only another document to collect dust on the bookshelf.

Universities—schools of education and mathematics departments in particular—have to be alert to the gaps and irrationalities of TSM and rededicate themselves to teaching SM to their future teachers.

Universities have not begun to take the issue of teacher education seriously.

The paradox is this:

- To provide this mathematical knowledge, we need the expertise of research mathematicians.

- Research mathematicians usually do not possess the needed knowledge of schools to get this job done.

- Research universities cannot afford to put the education of math teachers as a top priority.

Like all good things in life—freedom, for example—effective mathematics professional development is something we must fight for, everyday.

Good math teachers will materialize only when we are determined to negotiate a balance between highly charged conflicting demands.

Do we have the will to do it?