Equivalent Fractions and FFFP Allison Coates
Chapter 13

1.
$$\frac{161}{91} = \frac{7 * 23}{7 * 13} = \frac{23}{13}$$
$$\frac{253}{143} = \frac{11 * 23}{11 * 13} = \frac{23}{13}$$

Yes, both fractions are equivalent to $\frac{23}{13}$, and therefore, both fractions are equivalent to each other.

3.
$$\frac{9}{4} = \frac{5 \times 5 \times 9}{5 \times 5 \times 4} = \frac{225}{100} = 2.25$$

$$\frac{36}{125} = \frac{2^3 \times 36}{2^3 \times 5^3} = \frac{288}{1000} = .288$$

$$\frac{15}{8} = \frac{5^3 \times 15}{5^3 \times 2^3} = \frac{1875}{1000} = 1.875$$

4. To explain $\frac{6}{14} = \frac{3}{7}$ without using $\frac{m}{n} = \frac{cm}{cn}$ for all fractions $\frac{m}{n}$ and all nonzero c:

We begin by finding $\frac{6}{14}$ on the number line. We divide the unit segment (from 0 to 1) into 14 parts of equal length. Then, we tell our students "the fraction 6/14 is the length of the segment on the number line when we divide the unit from 0 to 1 into 14 parts of equal length and take the first 6."

Now tell the students "Now, where is $\frac{3}{7}$? We divide the unit segment into 7 parts of equal length. Then each new part of the seven is two of the 14 parts. The fraction $\frac{3}{7}$ is the length of the segment on the number line when we divide the unit into 7 parts of equal length, and take the first 3. Here, taking the first three of these seven gives us the exact same segment as taking the first 6 of 14. Since the two segments are the same, then the fractions must be the same. So they are equal.

Just as we can divide the unit from 0 to 1 into equal parts, we can divide any unit on the number line into equal parts. So we can divide up the unit from 1 to 2 (also of length 1), from 2 to 3, and so forth.

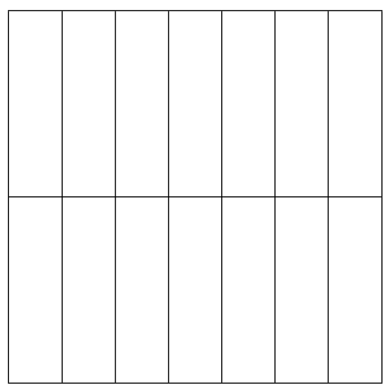
For $\frac{28}{24}$, divide the unit from 0 to 1 into 24 parts of equal length, and then divide the unit from 1 to 2 into 24 equal parts. Say "the fraction $\frac{28}{24}$ is the length of the segment on the number line when we divide each unit (the first from 0 to 1, the second from 1 to 2, etc.) into 24 equal parts and take the first 28, starting from 0. That means we take all of the first 24 (the whole unit from 0 to 1), and we take the first 4 of the second unit.

Now divide each of the units into 6 parts of equal length. Then each new part of the 6 is 4 of the 24 parts. The fraction $\frac{7}{6}$ is the length of the segment on the number line when we divide each of the units into 6 equal parts, and take the first 7. Here, taking the first seven gives us the exact same segment as taking the first 28 of 24. Since the two segments are the same, then the fractions must be the same. So they are equal.

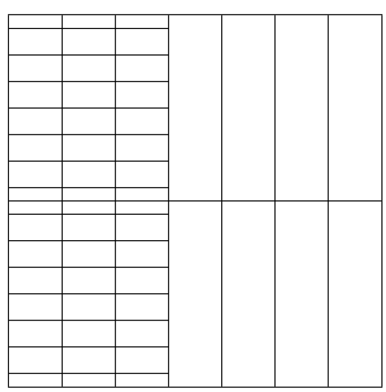
For $\frac{5}{2}$, we divide each unit of the number line into 2 equal parts. We say "the fraction $\frac{5}{2}$ is the length of the segment when we divide each unit into 2 equal parts and take the first five parts." We continue "Now imagine that instead of dividing the units into two equal pieces, we divided the units into 12 equal parts. Then the segment made from six new parts is the same as the segment made from one of the old parts. The length of the segment from 30 such new segments is the same as the length of the segment of 5 of the old segments. Since the segments are equal, then the fractions must be equal.

5. (a) We begin with the unit square. To represent the fraction $\frac{k}{l}$, we divide the unit square into l smaller rectangles, each of equal area. We will then represent the fraction $\frac{k}{l}$ as the total area of k of the l rectangles.

In the case of the fraction $\frac{6}{14}$, we create 14 rectangles as follows: divide the horizontal unit into 7 equal length segments, and divide the vertical unit into 2 equal length segments. Extending these divisions across the whole unit square creates 14 rectangles.



By construction, each rectangle is of the same width and length, so each rectangle as the same area. Therefore, each rectangle has area 1/14. Then the rectangle with sides $[0, 1/7]$ and $[0, 1/2]$ is of area 1/14, and the concatenation of 6 rectangles formed by the segment $[0, 3/7]$ and $[0, 1]$ represents $\frac{6}{14}$.



Now, to show how this is equal to $\frac{3}{7}$, we construct 7 equal rectangles in the unit square as follows: follows: divide the horizontal unit into 7 equal length segments, and divide the vertical unit into 1 equal length segment–that is, do not divide it. Extending these divisions across the whole unit square creates 7 rectangles. Byconstruction, each rectangle is of the same width and length, so each rectangle as the same area. Therefore, each rectangle has area 1/7. Then the rectangle with sides $[0, 1/7]$ and $[0, 1]$ is of area $\frac{1}{7}$, and the concatenation of 3 rectangles formed by the segment $[0, 3/7]$ and $[0, 1]$ represents $\frac{3}{7}$. But this is exactly the same area as constructed previously, so the fractions are the same.

(b) Following the proof of Theorem 13.1, we wish to prove that for any nonzero whole number c,

$$\frac{m}{n} = \frac{cm}{cn}.$$

We know that $\frac{m}{n}$ is m copies of $\frac{1}{n}$. We want to show that it is also cm copies of $\frac{1}{cn}$.
Using the area model, we have a unit square whose sides are of length 1. We divide the horizontal unit the n equal length segments. This partitions the unit square into n smaller rectangles, each of horizontal length $\frac{1}{n}$ and vertical length 1. Each rectangle has area $\frac{1}{n}$, and therefore, the rectangles are congruent.
The fraction $\frac{m}{n}$ then is m copies of the $\frac{1}{n}$ area rectangle.
Now observe that for each rectangle of area $\frac{1}{n}$, we could divide it into c smaller rectangles, by partitioning the vertical edge of the unit square into c segments. Then there are a total of $c \times n$ rectangles in the unit square, so each smaller rectangle has area $\frac{1}{cn}$. Then, each rectangle of area $\frac{1}{n}$ is c copies of $\frac{1}{cn}$. Since $\frac{m}{n}$ is m copies of the $\frac{1}{n}$ area rectangle, and the $\frac{1}{n}$ area rectangle is c copies of the $\frac{1}{cn}$ area rectangle, then $\frac{m}{n}$ is cm copies of the $\frac{1}{cn}$ area rectangle. But $\frac{cm}{cn}$ is also cm copies of the $\frac{1}{cn}$ area rectangle, so $\frac{m}{n} = \frac{cm}{cn}$.

8. (a) For which fraction $\frac{m}{n}$ is it true that $\frac{m}{n} = \frac{m+1}{n+1}$? By the Cross Multiplication Algorithm, $\frac{m}{n} = \frac{m+1}{n+1}$ is equivalent to $m(n+1) = n(m+1)$. That is,

$$\begin{aligned} m(n+1) &= n(m+1) \\ mn + m &= nm + n \\ m &= n \end{aligned}$$

So $\frac{m}{n} = \frac{m+1}{n+1}$ when $m = n$. When $m = n$, then $\frac{m}{n}$ is 1, the unit. $\frac{m}{n} = \frac{m+1}{n+1}$ when the fraction represented is the unit 1.

(b) For which fraction $\frac{m}{n}$ is it true that $\frac{m}{n} = \frac{m+b}{n+b}$?
$\frac{m}{n} = \frac{m+b}{n+b}$ when $m(n+b) = n(m+b)$, for positive whole number values of b. That is,

$$\begin{aligned} m(n+b) &= n(m+b) \\ mn + mb &= nm + nb \\ mb &= nb \\ m &= n \end{aligned}$$

So $\frac{m}{n} = \frac{m+b}{n+b}$ when $m = n$. As above, this equality holds for the unit fraction.

9. To show the equivalence of the following statements, we need to show that each statement can imply the other two.

We show that

$$\frac{a}{b} = \frac{c}{d} \implies \frac{a+b}{b} = \frac{c+d}{d}$$

as follows. First, assume $\frac{a}{b} = \frac{c}{d}$. Since they are equal, they will remain equal as we add 1 to both, and we express that 1 as $\frac{b}{b}$.

$$\begin{aligned} \frac{a}{b}+1 &= \frac{c}{d}+1 \\ \frac{a+b}{b} &= \frac{c}{d}+\frac{b}{b} \\ &= \frac{cb}{bd}+\frac{db}{bd} \text{ by FFFP} \\ &= \frac{cb+db}{bd} \\ &= \frac{c+d}{d} \end{aligned}$$

Now we show

$$\frac{a+b}{b} = \frac{c+d}{d} \implies \frac{a}{b} = \frac{c}{d}$$

$$\begin{aligned} \frac{a+b}{b} &= \frac{c+d}{d} \\ \frac{a}{b}+1 &= \frac{c}{d}+1 \\ \frac{a}{b} &= \frac{c}{d} \end{aligned}$$

So now we have shown the equivalence of statements (a) and (c). We must now show the equivalence of statements (a) and (b).

We show that

$$\frac{a}{b} = \frac{c}{d} \implies \frac{a}{a+b} = \frac{c}{c+d}$$

as follows. First, assume $\frac{a}{b} = \frac{c}{d}$. Then by the Cross Multiplication Algorithm, $ad = bc$ or $a = \frac{bc}{d}$.

$$\begin{aligned} a+b &= \frac{bc}{d}+b \\ &= \frac{bc}{d}+\frac{bd}{d} \\ &= \frac{b}{d}(c+d) \end{aligned}$$

Then

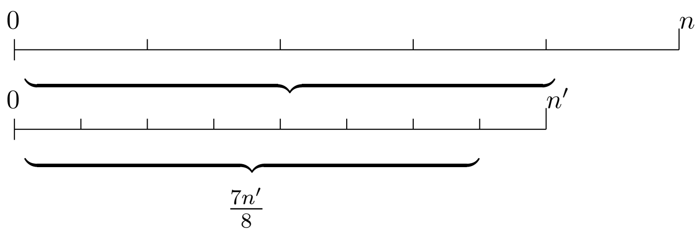
$$\frac{a}{a+b} = \frac{\frac{bc}{d}}{\frac{b}{d}(c+d)} = \frac{c}{c+d}.$$

So, statement (a) implies statement (b). Now we must show statement (b) implies statement (a). We assume $\frac{a}{a+b} = \frac{c}{c+d}$. Then by CMA,

$$\begin{aligned} a(c+d) &= c(a+b) \\ ac+ad &= ca+cb \\ ac+ad &= ac+cb \\ ad &= cb \end{aligned}$$

But again, by CMA, $ad = bc$ is equivalent to $\frac{a}{b} = \frac{c}{d}$. Therefore, statement (b) implies (a), and the equivalence is established.

11.



A flock of geese on a pond were observed continuously. At noon, $\frac{1}{5}$ of the geese flew away. At 1 pm, $\frac{1}{8}$ of the geese that remained flew away. Then 56 geese remained. How many were in the original flock?

We begin by labelling the unit on the number line from 0 to n, where n is the number of geese in the original flock. Then, we wish to determine the number of geese after $\frac{1}{5}$ of geese have left.

To determine the number of geese just after noon, we divide the unit into five pieces of equal length and take 4. We label this segment of 4 parts n'. At 1 pm, $\frac{1}{8}$ of n' geese leave. To represent this, we divide the new unit, from 0 to n' into 8 parts, and take 7. 56 geese remained when $\frac{7}{8}$ of the smaller amount of geese remained. Partitioning 56 into 7 equal parts, we see that $56 = 7 * 8$ so 1 part contains 8 geese. Therefore, 8 geese were in one part of the 8, so 64 geese must have been in the intermediate group. 64 geese remained when $\frac{4}{5}$ of the original geese remained. We break 64 into 4 equal parts, and take 1. Now, 16 geese are in 1 of the 5 parts of the original flock, so the original flock, which is the total of 5 parts, is 5*16, or there must have been 80 geese in the original flock.